

# POINTS OF CONTINUITY OF QUASICONVEX FUNCTIONS ON TOPOLOGICAL VECTOR SPACES

PATRICK J. RABIER

**ABSTRACT.** We give necessary and sufficient conditions for a real-valued quasiconvex function  $f$  on a Baire topological vector space  $X$  (in particular, Banach or Fréchet space) to be continuous at the points of a residual subset of  $X$ . These conditions involve only simple topological properties of the lower level sets of  $f$ . A main ingredient consists in taking advantage of a remarkable property of quasiconvex functions relative to a topological variant of essential extrema on the open subsets of  $X$ . One application is that if  $f$  is quasiconvex and continuous at the points of a residual subset of  $X$ , then with a single possible exception,  $f^{-1}(\alpha)$  is nowhere dense or has nonempty interior, as is the case for everywhere continuous functions. As a barely off-key complement, we also prove that every usc quasiconvex function is quasicontinuous in the (topological) sense of Kempisty since this interesting property does not seem to have been noticed before.

## 1. INTRODUCTION

If  $X$  is a vector space, a function  $f : X \rightarrow \overline{\mathbb{R}}$  is quasiconvex if its lower level sets

$$(1.1) \quad F_\alpha := \{x \in X : f(x) < \alpha\},$$

are convex for every  $\alpha \in \mathbb{R}$ . This is equivalent to the convexity of the level sets

$$(1.2) \quad F'_\alpha := \{x \in X : f(x) \leq \alpha\}$$

and also equivalent to assuming that  $f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}$  for every  $x, y \in X$  and  $\lambda \in [0, 1]$ .

It is a well known result of Crouzeix [8] (see also [6]) that a real-valued (i.e., finite) quasiconvex function  $f$  on  $\mathbb{R}^N$  is a.e. Fréchet differentiable and therefore a.e. continuous. That no property of this sort can generally be true when  $X$  is an infinite dimensional topological vector space (tvs) follows at once from the existence of discontinuous linear forms on  $X$ , which are nowhere continuous.

The features of the linear case extend to convex functions, which can only be nowhere or everywhere continuous, with the latter happening if and only if the function is bounded above on some nonempty open subset ([5, p. 92], [12]). Little seems to be known about the continuity of quasiconvex functions when  $\dim X = \infty$ . We are only aware of an unpublished report of Hadjisavvas [14], where another finite dimensional result of Crouzeix [9] is generalized; see [2] for more details.

The main goal of this paper is to show that in spite of sharp differences with the finite dimensional or convex case, there are still simple necessary and sufficient conditions for a real-valued quasiconvex function on a Baire tvs  $X$  to be continuous

---

1991 *Mathematics Subject Classification.* 26B05, 52A41, 54E52.

*Key words and phrases.* Quasiconvex function, topological essential extremum, point of continuity, point of discontinuity, Baire category, quasicontinuity.

at the points of a residual (and therefore dense) subset of  $X$ , i.e., discontinuous only at the points of a subset of (Baire) first category in  $X$ . These conditions involve only basic topological properties of the sublevel sets (1.1) and (1.2).

Above, “Baire tvs” refers to the fact that  $X$  is a tvs and that no nonempty open subset of  $X$  can be covered by countably many closed subsets with empty interior. Equivalently, no nonempty open subset of  $X$  is of first category. First and second categories are always understood relative to the whole space  $X$ .

Since every complete metric space is a Baire space, the class of Baire tvs includes Banach or Fréchet spaces, but there are many other non-metrizable examples in the literature. Local convexity is not required but significant applications may be limited without it. For example, if  $X = L^p(0, 1)$  with  $0 < p < 1$ , the only open convex subsets are  $X$  and  $\emptyset$  [7, p. 116], so that the quasiconvex functions on  $X$  continuous at the points of a residual subset are actually constant on a residual subset. This follows at once from the criterion in the next paragraph.

One of the main results, given in Corollary 5.3 along with other equivalent but slightly more technical statements, is that the set of points of discontinuity of a quasiconvex function  $f : X \rightarrow \mathbb{R}$  is of first category if and only if, with the notations (1.1) and (1.2),  $F_\alpha$  is nowhere dense whenever  $F'_\alpha = \emptyset$ . While the necessity is a little tricky but quickly proved, the path to sufficiency is longer and more meandering.

Before laying out our strategy to prove sufficiency, we point out that the above necessary and sufficient condition is satisfied (as it must be) if  $f$  is usc or lsc, or when  $X = \mathbb{R}^N$ . The latter yields a “category” variant of Crouzeix’s theorem without the differentiability part, which however can be given a straightforward direct proof (Remark 5.1).

A main ingredient that will come into play at the onset is the concept of “topological” essential extremum (Section 2), which mimics the familiar notion from measure theory. More specifically, if  $\mathcal{T}$  denotes the topology of  $X$  and  $U \subset X$  is an open<sup>1</sup> subset, we define

$$\mathcal{T} \operatorname{ess} \sup_U f := \inf \{ \alpha \in \mathbb{R} : \{f|_U > \alpha\} \text{ is of first category} \}$$

and, next,  $\mathcal{T} \operatorname{ess} \inf_U f := -\mathcal{T} \operatorname{ess} \sup_U (-f)$ , which of course is also given by a similar formula.

These  $\mathcal{T}$ -essential extrema are discussed in the next section, when  $X$  is a topological space. Although the aim there is just the proof of the elementary but basic Lemma 2.1, it seemed instructive to show that they are fairly versatile since special cases not only include the pointwise extrema, but also the classical essential extrema of measure theory, at least in the case of the Lebesgue measure.

Evidently, such a simple-minded and natural definition must have come to mind only moments after the discovery of the Baire categories<sup>2</sup>, but it appears to have remained rather confidential. A closely related idea is explored at some length in two 1963 papers by Semadeni [24], [25], for a very specific and different purpose. Curiously, this idea does not seem to resurface in other issues and we were unable to find any subsequent record of its use.

In this paper,  $\mathcal{T}$ -essential extrema are pervasive because  $\mathcal{T} \operatorname{ess} \inf_X f$  is a crucial value in the problem under investigation, but their importance is primarily due

<sup>1</sup>The openness of  $U$  is not important, but the definition will not be needed when  $U$  is not open.

<sup>2</sup>Figuratively speaking, since Baire category (1899) predates measure theory (1901).

to the combination of two features. First, by design, they are unchanged after modification of the function on a subset of first category. Second, quasiconvex functions have a remarkable property relative to these extrema: If  $f$  is quasiconvex (and  $U$  is open), then  $\mathcal{T} \text{ess sup}_U f = \sup_U f$ , the *pointwise* supremum of  $f$  on  $U$ , while  $\mathcal{T} \text{ess inf}_U f = \inf_U f$ , the *pointwise* infimum of  $f$  on  $U$ , provided that equality holds when  $U = X$  (Theorem 3.3). These properties are typical of lower and upper semicontinuous functions, respectively, but for quasiconvex functions, they hold without any semicontinuity requirement.

Next, if  $f$  and  $g$  are quasiconvex functions that agree on a residual subset of  $X$  (equivalent functions) and if  $x$  is a point of continuity of  $g$ , Theorem 3.3 is instrumental in discovering and proving a necessary and sufficient condition for  $x$  to be a point of continuity of  $f$  as well (Theorem 4.2). A corollary yields a necessary and sufficient condition for the set of points of discontinuity of  $f$  to be of first category if this is true of the set of points of discontinuity of  $g$  (Corollary 4.3).

In turn, this can be used to find a sharp sufficient condition for the set of points of discontinuity of a quasiconvex function  $f$  to be of first category, but now without having to know an equivalent quasiconvex function with that property. Indeed, under suitable assumptions about  $f$ , the usc hull of  $f$  is (quasiconvex and) real-valued and equivalent to  $f$ . Since the set of points of discontinuity of a semicontinuous function is of first category, the desired condition follows from the sufficiency part of Corollary 4.3 (Theorem 5.2). Ultimately, this leads to the sufficiency of the conditions listed in Corollary 5.3 and the main goal of the paper is achieved. For a rather surprising by-product, see Corollary 5.4.

In Section 6, we give two simple sufficient (not necessary) conditions for the set of points of discontinuity of a quasiconvex function to be of first category. One of them (Theorem 6.1) is used to show that if the set of points of discontinuity of a quasiconvex function  $f$  is of first category, then with only one possible exception,  $f^{-1}(\alpha)$  is either nowhere dense or has nonempty interior (Theorem 6.3). Even if  $f$  is usc or lsc, the proof requires using Theorem 6.1 in the nontrivial case when the function of interest is not semicontinuous. We also show that  $f$  and its lsc hull have the same points of continuity (Theorem 6.4), which is false when the set of points of discontinuity of  $f$  is of second category.

The last section is devoted to two complements of independent interest. We prove that the sets  $\overset{\circ}{F}_\alpha$  and  $\overset{\circ}{F}'_\alpha$  (see (1.1) and (1.2)), which play a key role in the paper, are unchanged when the quasiconvex function  $f$  is replaced by an equivalent quasiconvex function (Theorem 7.2). This clarifies the connection between various results in the preceding sections.

Lastly, we prove that every usc quasiconvex function is quasicontinuous in the (classical) sense of Kempisty. This was suggested by some technical conditions related to the proof of Theorem 3.3 and seems to have so far remained unnoticed, even when  $X = \mathbb{R}^N$ .

## 2. TOPOLOGICAL ESSENTIAL EXTREMA

As mentioned in the Introduction, if  $X$  is a topological space with topology  $\mathcal{T}$ ,  $f : X \rightarrow \overline{\mathbb{R}}$  is a given function and  $U \subset X$  is open and nonempty, we set

$$(2.1) \quad \mathcal{T} \operatorname{ess} \sup_U f := \inf \{ \alpha \in \mathbb{R} : \{f|_U > \alpha\} \text{ is of first category} \} = \\ \sup \{ \alpha \in \mathbb{R} : \{f|_U > \alpha\} \text{ is of second category} \}.$$

The second equality in (2.1) follows from the fact that the collection of upper level sets  $(\{f|_U > \alpha\})_{\alpha \in \mathbb{R}}$  is nonincreasing and, if convenient, the level sets  $\{f|_U > \alpha\}$  can be replaced by  $\{f|_U \geq \alpha\}$  without prejudice. If  $U = \emptyset$ , we set  $\mathcal{T} \operatorname{ess} \sup_{\emptyset} f := -\infty$ . It is always true that  $\mathcal{T} \operatorname{ess} \sup_U f \leq \sup_U f$  and, if  $X$  is a Baire space and  $f$  is continuous, then  $\mathcal{T} \operatorname{ess} \sup_U f = \sup_U f$ .

When  $\mathcal{T}$  is the discrete topology on a set  $X$ , (2.1) coincides with  $\sup_U f$  since  $\emptyset$  is the only subset of first category. It is less obvious but shown in the example below that the definition (2.1) also incorporates the classical essential supremum of a Lebesgue measurable function as a special case.

**Example 2.1.** Let  $X = \mathbb{R}^N$  and  $\mathcal{T} = \mathcal{D}$ , the density topology, whose open subsets are  $\emptyset$  and the Lebesgue measurable subsets with density 1 at each of their points ([13], [15], [18]). With this topology,  $\mathbb{R}^N$  is not a metric space ( $\mathcal{D}$  is not first countable) and not even a *tus*, but it is a Baire space. If  $f$  is measurable and  $S \subset \mathbb{R}^N$  is a measurable subset with density interior  $U$ , then  $\mathcal{D} \operatorname{ess} \sup_U f = \operatorname{ess} \sup_S f$ : First,  $\operatorname{ess} \sup_S f = \operatorname{ess} \sup_U f$  because  $U$  is the set of points at which  $S$  has density 1, so that  $S \setminus U$  is a null set (set of measure zero) by the Lebesgue density theorem. Next,  $\operatorname{ess} \sup_U f = \mathcal{D} \operatorname{ess} \sup_U f$  because a subset is of first category for  $\mathcal{D}$  if and only if it is a null set.

In addition, (2.1) makes sense when  $f$  is not measurable. For example, if  $A \subset X$  is not measurable and  $f = \chi_A$ , then  $\mathcal{D} \operatorname{ess} \sup_{\mathbb{R}^N} \chi_A = 1$  (if  $A$  were of first category for  $\mathcal{D}$ , it would be a null set and hence measurable). As odd as it might seem, there is actually nothing new here: Although rarely if ever used, the definition of  $\operatorname{ess} \sup_U f$  makes sense for non-measurable  $f$  by using the Lebesgue outer measure  $\mu_N^*$ , viz.,  $\operatorname{ess} \sup_U f = \inf \{ \alpha \in \mathbb{R} : \mu_N^*(\{f|_U > \alpha\}) = 0 \}$ . Since  $\mu_N^*$  vanishes exactly on null sets and the null sets are the subsets of first category for  $\mathcal{D}$ , this is just  $\mathcal{D} \operatorname{ess} \sup_U f$ .

Naturally, we can also define  $\mathcal{T} \operatorname{ess} \inf_U f = -\mathcal{T} \operatorname{ess} \sup_U (-f)$ , that is,

$$(2.2) \quad \mathcal{T} \operatorname{ess} \inf_U f := \sup \{ \alpha \in \mathbb{R} : \{f|_U < \alpha\} \text{ is of first category} \} = \\ \inf \{ \alpha \in \mathbb{R} : \{f|_U < \alpha\} \text{ is of second category} \},$$

when  $U$  is open and nonempty and  $\mathcal{T} \operatorname{ess} \inf_{\emptyset} f := \infty$ . Once again, the level sets  $\{f|_U \leq \alpha\}$  can be used instead of  $\{f|_U < \alpha\}$  and  $\inf_U f \leq \mathcal{T} \operatorname{ess} \inf_U f$ , with equality (for instance) when  $X$  is a Baire space and  $f$  is continuous.

If  $f$  and  $g$  are two extended real valued functions on  $X$ , it is natural to define the  $\mathcal{T}$ -equivalence of  $f$  and  $g$ ,  $f \sim_{\mathcal{T}} g$  for short, by

$$(2.3) \quad f \sim_{\mathcal{T}} g \text{ if } f = g \text{ on a residual subset of } X,$$

where, as usual, a residual subset is the complement of a set of first category. Thus,  $f \sim_{\mathcal{T}} g$  if and only if  $f \neq g$  on a set of first category. Since the union of two sets

of first category is of first category,

$$(2.4) \quad f \sim_{\mathcal{T}} g \Rightarrow \mathcal{T} \operatorname{ess} \sup_U f = \mathcal{T} \operatorname{ess} \sup_U g \text{ and } \mathcal{T} \operatorname{ess} \inf_U f = \mathcal{T} \operatorname{ess} \inf_U g,$$

for every open subset  $U \subset X$ .

The question whether the pointwise and  $\mathcal{T}$ -essential extrema of a real-valued function coincide on every open subset will be of central importance in the next section. In turn, this will hinge on the following simple lemma.

**Lemma 2.1.** *Let  $X$  be a topological space with topology  $\mathcal{T}$ . For every function  $f : X \rightarrow \mathbb{R}$ , the following statements are equivalent.*

- (i)  $\sup_U f = \mathcal{T} \operatorname{ess} \sup_U f$  for every open subset  $U \subset X$ .
- (ii) For every  $x_0 \in X$ , every open subset  $U \subset X$  containing  $x_0$  and every  $\varepsilon > 0$ , the set  $\{x \in U : f(x) > f(x_0) - \varepsilon\}$  is of second category.

*Likewise, the following statements are equivalent:*

- (i')  $\inf_U f = \mathcal{T} \operatorname{ess} \inf_U f$  for every open subset  $U \subset X$ .
- (ii') For every  $x_0 \in X$ , every open subset  $U \subset X$  containing  $x_0$  and every  $\varepsilon > 0$ , the set  $\{x \in U : f(x) < f(x_0) + \varepsilon\}$  is of second category.

*Proof.* (i)  $\Rightarrow$  (ii). Suppose that (i) holds and, by contradiction, assume that there are  $x_0 \in X$ , an open subset  $U \subset X$  containing  $x_0$  and some  $\varepsilon > 0$  such that  $\{x \in U : f(x) > f(x_0) - \varepsilon\}$  is of first category. Then,  $\mathcal{T} \operatorname{ess} \sup_U f \leq f(x_0) - \varepsilon < f(x_0) \leq \sup_U f$ , which contradicts (i).

(ii)  $\Rightarrow$  (i). Let  $U \subset X$  be an open subset. We argue by contradiction, thereby assuming that  $\sup_U f > \mathcal{T} \operatorname{ess} \sup_U f$ . If so,  $U$  is not empty (otherwise, both suprema are  $-\infty$ ) and  $\mathcal{T} \operatorname{ess} \sup_U f < \infty$ . Thus, the assumption  $\sup_U f > \mathcal{T} \operatorname{ess} \sup_U f$  implies the existence of  $x_0 \in X$  such that  $\mathcal{T} \operatorname{ess} \sup_U f < f(x_0) \leq \sup_U f$ . Choose  $\varepsilon > 0$  small enough that  $\mathcal{T} \operatorname{ess} \sup_U f < f(x_0) - \varepsilon$ . By (ii),  $\{x \in U : f(x) > f(x_0) - \varepsilon\}$  is of second category, so that  $\mathcal{T} \operatorname{ess} \sup_U f \geq f(x_0) - \varepsilon$ , which is a contradiction.

That (i')  $\Leftrightarrow$  (ii') follows by replacing  $f$  by  $-f$  above.  $\square$

### 3. EXTREMA OF QUASICONVEX FUNCTIONS ON OPEN SUBSETS

From this point on,  $X$  is a Baire tvs with topology  $\mathcal{T}$ . We shall show that under a simple necessary and sufficient condition on the quasiconvex function  $f$ , the  $\mathcal{T}$ -essential extrema of  $f$  on any open subset  $U \subset X$  coincide with its corresponding pointwise extrema on  $U$ . This will follow from Lemma 2.1 after we expose some properties of convex sets relative to Baire category (Lemma 3.2 below).

Convex subsets of infinite dimensional spaces have less pleasant features than their finite dimensional counterparts. However, an important one remains unchanged, which is spelled out in the following remark.

**Remark 3.1.** *If  $C \subset X$  is a convex subset with  $\overset{\circ}{C} \neq \emptyset$ , then  $\overset{\circ}{C}$  is convex, the closures of  $C$  and  $\overset{\circ}{C}$  are the same (i.e.,  $\overline{C}$ ) and the interior of  $C$  and  $\overline{C}$  are the same (i.e.,  $\overset{\circ}{C}$ ) [3, p.105]. In particular,  $\partial C = \partial \overset{\circ}{C} = \partial \overline{C}$ . When  $\dim X = \infty$ , all these properties break down when  $\overset{\circ}{C} = \emptyset$ . However,  $\overline{C}$  is always convex [3, p. 103].*

The next lemma will be used several times, including in the proof of part (ii) of Lemma 3.2 below.

**Lemma 3.1.** *Let  $U$  and  $C$  be subsets of  $X$  with  $U$  open and  $C$  convex. If  $A \subset X$  is of first category and  $U \setminus A \subset C$ , then  $U \subset C$ .*

*Proof.* It suffices to show that if  $x_0 \in U \cap A \neq \emptyset$ , then  $x_0 \in C$ . After translation, it is not restrictive to assume that  $x_0 = 0$ .

The set  $V := U \cap (-U) \subset U$  is an open neighborhood of 0 and  $A \cup (-A)$  is of first category. Thus,  $V \setminus (A \cup (-A)) \neq \emptyset$ . If  $x \in V \setminus (A \cup (-A))$ , both  $x$  and  $-x$  are in  $V \setminus A \subset U \setminus A$ , hence in  $C$ . Since  $C$  is convex,  $0 = \frac{1}{2}x + \frac{1}{2}(-x) \in C$ .  $\square$

**Lemma 3.2.** *Let  $C \subset X$  be a convex subset and  $U \subset X$  be an open subset.*

- (i) *If  $C$  is of second category and  $U \cap C \neq \emptyset$ , then  $U \cap C$  is of second category.*
- (ii) *If  $U \cap (X \setminus C) \neq \emptyset$ , then  $U \cap (X \setminus C)$  is of second category.*

*Proof.* (i) By contradiction, assume that  $U \cap C$  is of first category. Since  $U \cap C \neq \emptyset$ , pick  $x_0 \in U \cap C$ . After translation, it is not restrictive to assume  $x_0 = 0 \in U \cap C$ . Let  $x \in C$  be given. Then,  $\frac{1}{n}x \in C$  for every  $n \in \mathbb{N}$  and  $\frac{1}{n}x \in U$  if  $n$  is large enough since  $U$  is open and the scalar multiplication is continuous. Thus,  $\frac{1}{n}x \in U \cap C$  for every  $x \in C$  and some  $n \in \mathbb{N}$ . This means that  $C \subset \cup_{n \in \mathbb{N}} n(U \cap C)$ . Since  $U \cap C$  is of first category, the same thing is true of  $n(U \cap C)$  for every  $n$ , so that  $\cup_{n \in \mathbb{N}} n(U \cap C)$  is of first category. But then,  $C$  is of first category, which is a contradiction.

(ii) Since  $U \cap (X \setminus C) = U \setminus C \neq \emptyset$ , it is obvious that  $U \not\subset C$ . By contradiction, assume that  $U \setminus C$  is of first category and note that  $U \setminus (U \setminus C) = U \cap C \subset C$ . Since  $U$  is open,  $C$  is convex and  $U \setminus C$  is of first category, it follows from Lemma 3.1 that  $U \subset C$ , which is a contradiction.  $\square$

Given a function  $f : X \rightarrow \mathbb{R}$  and  $\alpha \in \mathbb{R}$ , recall the notation  $F_\alpha := \{x \in X : f(x) < \alpha\}$  and  $F'_\alpha := \{x \in X : f(x) \leq \alpha\}$  in (1.1) and (1.2). This notation will only be used without further specification when the function of interest is called  $f$ .

**Theorem 3.3.** *Let  $f : X \rightarrow \overline{\mathbb{R}}$  be quasiconvex.*

- (i)  $\sup_U f = \mathcal{T} \text{ess sup}_U f$  for every open subset  $U \subset \mathbb{R}^N$ .
- (ii)  $\inf_U f = \mathcal{T} \text{ess inf}_U f$  for every open subset  $U \subset X$  if and only if this is true when  $U = X$ .

*Proof.* Since changing  $f$  into  $\arctan f$  does not affect quasiconvexity and since it is readily checked that  $\arctan$  commutes with  $\mathcal{T} \text{ess sup}_U$  and  $\mathcal{T} \text{ess inf}_U$  (as well as with  $\sup_U$  and  $\inf_U$ ), we assume with no loss of generality that  $f$  is finite.

(i) We show that the condition (ii) of Lemma 2.1 holds and use its equivalence with part (i) of that lemma.

Pick  $x_0 \in X$  along with an open subset  $U \subset X$  containing  $x_0$  and  $\varepsilon > 0$ . The set  $\{x \in U : f(x) > f(x_0) - \varepsilon\}$  contains  $x_0$  and is the (nonempty) intersection  $U \cap (X \setminus C)$  where  $C := F'_{f(x_0) - \varepsilon}$  is convex. That  $U \cap (X \setminus C)$  is of second category follows from part (ii) of Lemma 3.2.

(ii) It is obvious that  $\inf_X f = \mathcal{T} \text{ess inf}_X f$  is necessary. Conversely, assuming this, we show that condition (ii') of Lemma 2.1 holds and use its equivalence with part (i') of that lemma.

Pick  $x_0 \in X$  along with an open subset  $U \subset X$  containing  $x_0$  and  $\varepsilon > 0$ . The set  $\{x \in U : f(x) < f(x_0) + \varepsilon\}$  contains  $x_0$  and is the (nonempty) intersection  $U \cap C$  where  $C := F_{f(x_0) + \varepsilon}$  is convex. It is also of second category by definition of  $\mathcal{T} \text{ess inf}_X f$  since  $\mathcal{T} \text{ess inf}_X f = \inf_X f \leq f(x_0) < f(x_0) + \varepsilon$ . That  $U \cap C$  is of second category follows from part (i) of Lemma 3.2.  $\square$

## 4. POINTS OF CONTINUITY OF EQUIVALENT QUASICONVEX FUNCTIONS

Suppose that  $f, g : X \rightarrow \mathbb{R}$  are quasiconvex functions and that  $f \sim_{\mathcal{T}} g$  (see (2.3)). In this section, we give a complete answer to the following question: If  $x$  is a point of continuity of one function, when is  $x$  a point of continuity of the other? That such a question can be answered is due to the properties highlighted in Theorem 3.3, combined with the quasiconvexity of both functions.

We confine attention to real-valued functions, because defining points of continuity is possible, but unnecessarily convoluted, for extended real-valued functions. The issue is that since the *size* of such sets is the matter of interest, it would not be adequate to require, as is normally done, that the function be finite at a point of continuity. Instead, it is much simpler to use the usual arctan trick to reduce the problem to the real-valued case.

We shall implicitly use the elementary remark that if  $f$  is any real-valued function on  $X$ , then  $\mathcal{T} \text{ess inf}_X f < \infty$ , for otherwise  $F_n$  is of first category for every  $n \in \mathbb{N}$  and so  $X = \bigcup_{n \in \mathbb{N}} F_n$  is of first category, which is absurd since  $X$  is a Baire space. As a result, we will never have to be concerned that any  $\mathcal{T}$ -essential infimum might be  $\infty$ . They can, of course, be  $-\infty$ . We begin with a special case.

**Lemma 4.1.** *Let  $f, g : X \rightarrow \mathbb{R}$  be quasiconvex functions such that  $f \sim_{\mathcal{T}} g$ , so that  $\mathcal{T} \text{ess inf}_X f = \mathcal{T} \text{ess inf}_X g := m \geq -\infty$  (see (2.4)).*

*(i) If also  $\inf_X f = m = \inf_X g$  (always true if  $m = -\infty$ ), then  $f$  and  $g$  have the same points of continuity and achieve a common value at such points.*

*(ii)  $\max\{f, m\}$  and  $\max\{g, m\}$  have the same points of continuity and achieve a common value at such points.*

*Proof.* (i) Let  $x$  denote a point of continuity of  $f$ , so that for every  $\varepsilon > 0$ , there is an open neighborhood  $U$  of  $x$  such that  $f(U) \subset [f(x) - \varepsilon, f(x) + \varepsilon]$ . Thus,  $\inf_U f \geq f(x) - \varepsilon$  and  $\sup_U f \leq f(x) + \varepsilon$ . By Theorem 3.3, this is the same as  $\mathcal{T} \text{ess inf}_U f \geq f(x) - \varepsilon$  and  $\mathcal{T} \text{ess sup}_U f \leq f(x) + \varepsilon$ .

Since  $f = g$  a.e., the  $\mathcal{T}$ -essential extremum is unchanged when  $f$  is replaced by  $g$ , so that  $\mathcal{T} \text{ess inf}_U g \geq f(x) - \varepsilon$  and  $\mathcal{T} \text{ess sup}_U g \leq f(x) + \varepsilon$ . By using once again Theorem 3.3, it follows that  $\inf_U g \geq f(x) - \varepsilon$  and  $\sup_U g \leq f(x) + \varepsilon$ , whence  $g(U) \subset [f(x) - \varepsilon, f(x) + \varepsilon]$ . In particular,  $g(x) \in [f(x) - \varepsilon, f(x) + \varepsilon]$ . Since  $\varepsilon > 0$  is arbitrary, it follows that  $g(x) = f(x)$  and hence that  $g(U) \subset [g(x) - \varepsilon, g(x) + \varepsilon]$ , which proves the continuity of  $g$  at  $x$ .

In summary, the points of continuity of  $f$  are points of continuity of  $g$  and  $g = f$  at such points. By exchanging the roles of  $f$  and  $g$ , the converse is true.

(ii) Just use (i) with  $\max\{f, m\}$  and  $\max\{g, m\}$ , respectively. Quasiconvexity is not affected and  $\max\{f, m\} \sim_{\mathcal{T}} \max\{g, m\}$  while  $\mathcal{T} \text{ess inf}_X \max\{f, m\} = \mathcal{T} \text{ess inf}_X \max\{g, m\} = m$ , so that  $\inf_X \max\{f, m\} = m = \inf_X \max\{g, m\}$  follows from  $\max\{f, m\} \geq m, \max\{g, m\} \geq m$  and  $\inf_X \leq \mathcal{T} \text{ess inf}_X$ .  $\square$

The next example shows that the condition  $\inf_X f = m = \inf_X g$  cannot be dropped in part (i) of Lemma 4.1.

**Example 4.1.** *Let  $X$  be infinite dimensional and let  $H \subset X$  be a dense hyperplane of first category. It was first shown by Arias de Reyna [1] in 1980 that such a hyperplane exists when  $X$  is a separable Banach space, but they also exist in general (Saxon [23]; all known proofs, including [1], assume a weak form of the Continuum Hypothesis). If  $f = \chi_{X \setminus H}$  and  $g = 1$ , then both  $f$  and  $g$  are quasiconvex and  $f \sim_{\mathcal{T}} g$ .*

However,  $\inf_X f = 0 \neq m = 1 = \inf_X g$  and indeed  $f$  is nowhere continuous while  $g$  is everywhere continuous.

**Theorem 4.2.** *Let  $f, g : X \rightarrow \mathbb{R}$  be quasiconvex functions such that  $f \sim_{\mathcal{T}} g$ , so that  $\mathcal{T} \operatorname{ess} \inf_X f = \mathcal{T} \operatorname{ess} \inf_X g := m \geq -\infty$ . If  $x$  is a point of continuity of  $g$ , then  $g(x) \geq m$ . Furthermore, a point of continuity  $x$  of  $g$  is not one of  $f$  if and only if  $m > -\infty, g(x) = m$  and  $x \in F'_m \cap (\cup_{\alpha < m} \overline{F}_\alpha)$ .*

*Proof.* If  $m = -\infty$ , the only nontrivial property follows from part (i) of Lemma 4.1. Below,  $m > -\infty$  and  $x$  denotes a point of continuity of  $g$ . To make the proof more transparent, we give four preliminary results.

(i)  $g(x) \geq m$ . By contradiction, assume that  $g(x) < m$  and let  $\alpha \in \mathbb{R}$  be such that  $g(x) < \alpha < m$ . By definition of  $m$ , the set  $G_\alpha := \{y \in X : g(y) < \alpha\}$  is of first category. On the other hand,  $G_\alpha = g^{-1}((-\infty, \alpha))$  is a neighborhood of  $x$  in  $X$  since  $g$  is continuous at  $x$ . Therefore,  $G_\alpha$  is of second category, which is a contradiction.

(ii)  $x \notin \cup_{\alpha < m} \overline{G}_\alpha$ . By (i),  $g(x) \geq m$ . By contradiction, assume that  $x \in \overline{G}_\alpha$  for some  $\alpha < m$ , whence  $U \cap G_\alpha \neq \emptyset$  for every neighborhood  $U$  of  $x$ . Choose  $\varepsilon > 0$  such that  $\alpha < g(x) - \varepsilon$ . Since  $g$  is continuous at  $x$ , then  $U := g^{-1}((g(x) - \varepsilon, \infty))$  is a neighborhood of  $x$  but  $U \cap G_\alpha = \emptyset$  and a contradiction is reached.

(iii) If  $g(x) > m$  or  $f(x) > m$ , then  $f$  is continuous at  $x$ . Indeed,  $x$  is a point of continuity of  $\max\{g, m\}$ . Thus, by part (ii) of Lemma 4.1, it is a point of continuity of  $\max\{f, m\}$  and  $\max\{g(x), m\} = \max\{f(x), m\}$ . As a result,  $g(x) > m$  and  $f(x) > m$ . Let  $\varepsilon > 0$  be small enough that  $m < f(x) - \varepsilon$  and let  $I_\varepsilon := (f(x) - \varepsilon, f(x) + \varepsilon)$ . Then,  $(\max\{f, m\})^{-1}(I_\varepsilon)$  is a neighborhood  $W_\varepsilon$  of  $x$ . From the choice of  $\varepsilon$ , it is obvious that  $W_\varepsilon = f^{-1}(I_\varepsilon)$ . Since this is true for every  $\varepsilon > 0$  small enough, it follows that  $f$  is continuous at  $x$ .

(iv) If  $f(x) = m$  and  $x \notin \cup_{\alpha < m} \overline{F}_\alpha$ , then  $f$  is continuous at  $x$ . Recall that  $x$  is a point of continuity of  $\max\{f, m\}$  (see (iii)). Accordingly, given  $\varepsilon > 0$ , the set  $V := (\max\{f, m\})^{-1}((-\infty, m + \varepsilon))$  is a neighborhood of  $x$  and, if  $y \in V$ , then  $f(y) < m + \varepsilon$ . On the other hand,  $x \notin \overline{F}_{m - \frac{\varepsilon}{2}}$ , so that there is a neighborhood  $W$  of  $x$  such that  $W \cap F_{m - \frac{\varepsilon}{2}} = \emptyset$ . Equivalently,  $f(y) \geq m - \frac{\varepsilon}{2}$  for  $y \in W$  and so, if  $y \in V \cap W$ , a neighborhood of  $x$ , then  $m - \varepsilon < f(y) < m + \varepsilon$ . Since  $f(x) = m$ , this reads  $|f(y) - f(x)| < \varepsilon$ , which shows that  $f$  is continuous at  $x$ .

We now prove the theorem. That  $g(x) \geq m$  is (i). Next, if  $x$  is not a point of continuity of  $f$ , then  $g(x) = m$  and  $f(x) \leq m$ , i.e.,  $x \in F'_m$ , by (i) and (iii). That  $x \in \cup_{\alpha < m} \overline{F}_\alpha$  is obvious if  $x \in F_m = \cup_{\alpha < m} F_\alpha$  and so, since  $x \in F'_m$  is already known, it suffices to show that  $x \in \cup_{\alpha < m} \overline{F}_\alpha$  if  $f(x) = m$ . This follows from (iv) since  $x$  is not a point of continuity of  $f$ . This proves the necessity of the conditions. By (ii) for  $f$  instead of  $g$  (i.e., if  $x \in \cup_{\alpha < m} \overline{F}_\alpha$ , then  $f$  is not continuous at  $x$ ), their sufficiency is obvious.  $\square$

With  $f$  and  $g$  of Example 4.1,  $g = m = 1$  is constant but  $\overline{F}_\alpha = X$  for every  $\alpha \in (0, 1)$  and  $F'_1 = X$  so that, by Theorem 4.2,  $f$  is nowhere continuous, which is indeed true. If the hyperplane  $H$  in that example is of second category (such hyperplanes are easily constructed in any infinite dimensional tvs [27, p. 14]), then Theorem 4.2 is not applicable since  $f \neq g$  on  $H$ , of second category.

**Corollary 4.3.** *Let  $f, g : X \rightarrow \mathbb{R}$  be quasiconvex with  $f \sim_{\mathcal{T}} g$ , so that  $\mathcal{T} \operatorname{ess} \inf_X f = \mathcal{T} \operatorname{ess} \inf_X g := m \geq -\infty$ . Suppose also that the set of points of discontinuity of  $g$  is of first category. Then, the set of points of discontinuity of  $f$  is of first category if and only if one of the following two conditions holds:*



- (i)  $m = -\infty$ .
- (ii)  $m > -\infty$  and  $f^{-1}(m) \cap \overline{F}_\alpha$  is of first category for every  $\alpha < m$ .

*Proof.* If  $x$  is a point of discontinuity of  $f$ , it is either a point of discontinuity of both  $f$  and  $g$ , or a point of continuity of  $g$  which is not one of  $f$ . Since the set  $B$  of points of discontinuity of  $g$  is of first category, the same thing is true of the set of points of discontinuity of  $f$  if and only if the points of  $X \setminus B$  (i.e., points of continuity of  $g$ ) that are not points of continuity of  $f$  belong in a set of first category. By Theorem 4.2, this happens if and only if (1)  $m = -\infty$  (and then every point of  $X \setminus B$  is a point of continuity of  $f$ ) or (2)  $m > -\infty$  and  $\{x \in F'_m \cap (\cup_{\alpha < m} \overline{F}_\alpha) : g(x) = m, x \in X \setminus B\}$  is of first category. For clarity, rewrite this set as  $(g^{-1}(m) \setminus B) \cap F$  where  $F := F'_m \cap (\cup_{\alpha < m} \overline{F}_\alpha)$ . Since  $B$  is of first category and  $(g^{-1}(m) \setminus B) \cap F = ((g^{-1}(m) \cap F) \setminus B)$ , the sets  $(g^{-1}(m) \setminus B) \cap F$  and  $g^{-1}(m) \cap F$  are of first category simultaneously. Furthermore, since  $f \sim_{\mathcal{T}} g$ , the sets  $g^{-1}(m) \cap F$  and  $f^{-1}(m) \cap F$  are also of first category simultaneously. But  $f^{-1}(m) \subset F'_m$ , so that  $f^{-1}(m) \cap F$  is just  $f^{-1}(m) \cap (\cup_{\alpha < m} \overline{F}_\alpha)$ . Since the union is actually a countable union, this set is of first category if and only if  $f^{-1}(m) \cap \overline{F}_\alpha$  is of first category for every  $\alpha < m$ .  $\square$

In Example 4.1,  $m = 1$ ,  $\overline{F}_\alpha = X$  if  $\alpha \in (0, 1)$  and  $f^{-1}(1)$  is residual (hence of second category since  $X$  is a Baire space) so that condition (ii) of Corollary 4.3 fails.

## 5. CONTINUITY AT THE POINTS OF A RESIDUAL SUBSET

We shall give a few equivalent necessary and sufficient conditions for the set of points of discontinuity of a quasiconvex function  $f : X \rightarrow \mathbb{R}$  to be of first category (Corollary 5.3). We begin with the construction of an auxiliary function.

**Lemma 5.1.** *Let  $f : X \rightarrow \mathbb{R}$  be quasiconvex and set*

$$(5.1) \quad \overline{f}(x) := \inf\{\alpha \in \mathbb{R} : x \in \overset{\circ}{F}_\alpha\}.$$

*The following properties hold:*

- (i)  $f \leq \overline{f}$ .
- (ii)  $\overline{f}$  is quasiconvex and upper semicontinuous (usc).
- (iii) If, in addition,  $\overset{\circ}{F}_\alpha \neq \emptyset$  whenever  $\alpha > m := \mathcal{T} \text{ess inf}_X f$ , then  $\overline{f}$  is real-valued and  $f \sim_{\mathcal{T}} \overline{f}$ .

*Proof.* (i) By contradiction, if  $x \in X$  and  $\overline{f}(x) < f(x)$ , there is  $\alpha < f(x)$  such that  $x \in \overset{\circ}{F}_\alpha \subset F_\alpha$  and so  $f(x) < \alpha < f(x)$ , which is absurd.

(ii) Both properties follow from the simple remark that, if  $\beta \in \mathbb{R}$ , then  $\{x \in X : \overline{f}(x) < \beta\} = \cup_{\alpha < \beta} \overset{\circ}{F}_\alpha$ , an open convex subset since the sets  $\overset{\circ}{F}_\alpha$  are convex and linearly ordered.

(iii) By (i),  $\overline{f}(x) > -\infty$  since  $f$  is real-valued. It remains to show that if it is assumed that  $\overset{\circ}{F}_\alpha \neq \emptyset$  when  $\alpha > m$ , then  $\overline{f}(x) < \infty$  and  $f \sim_{\mathcal{T}} \overline{f}$ . From now on, we set  $\mathbb{Q}_m := \mathbb{Q} \setminus \{m\}$ , a countable dense subset of  $\mathbb{R}$ . Of course,  $\mathbb{Q}_m = \mathbb{Q}$  if  $m = -\infty$  or  $m$  is irrational.

By definition of  $\overline{f}$  in (5.1),  $\overline{f}(x) < \infty$  for every  $x \in X$  if (and in fact only if)  $C := \cup_{\alpha \in \mathbb{Q}_m} \overset{\circ}{F}_\alpha = X$ . By contradiction, assume  $C \neq X$ . Since  $C$  is convex, it follows

from part (ii) of Lemma 3.2 with  $U = X$  that  $X \setminus C$  is of second category. Now, if  $x \in X \setminus C$ , there is  $\alpha \in \mathbb{Q}_m$  such that  $\alpha > \max\{m, f(x)\}$  (recall that  $m = \infty$  does not occur), so that  $x \in F_\alpha$ . Actually,  $x \in \partial F_\alpha$  since  $x \notin \overset{\circ}{F}_\alpha \subset C$  and  $\partial F_\alpha = \partial \overset{\circ}{F}_\alpha$  by Remark 3.1 and the standing assumption that  $\overset{\circ}{F}_\alpha \neq \emptyset$  when  $\alpha > m$ . Thus,  $\partial F_\alpha$  is nowhere dense (recall that boundaries of open subsets are nowhere dense). As a result,  $X \setminus C \subset \bigcup_{\alpha \in \mathbb{Q}_m, \alpha > m} \partial F_\alpha$ , a set of first category since  $\mathbb{Q}_m$  is countable. Thus,  $X \setminus C$  is of first category. This contradiction shows that  $C = X$ .

To complete the proof, we show that  $f \sim_{\mathcal{T}} \bar{f}$ . Set  $\Sigma := \{x \in X : f(x) \neq \bar{f}(x)\} = \{x \in X : f(x) < \bar{f}(x)\}$  by part (i). By the denseness of  $\mathbb{Q}_m$  in  $\mathbb{R}$ ,  $\Sigma = \bigcup_{\alpha \in \mathbb{Q}_m} \Sigma(\alpha)$  where  $\Sigma(\alpha) := \{x \in X : f(x) < \alpha < \bar{f}(x)\}$ . Since  $\mathbb{Q}_m$  is countable, it suffices to show that  $\Sigma(\alpha)$  is of first category for every  $\alpha \in \mathbb{Q}_m$ .

If  $x \in \Sigma(\alpha)$ , then  $x \in F_\alpha$  and, if  $\alpha < m$ , it follows from the definition of  $m$  that  $F_\alpha$  is of first category. If now  $\alpha > m$ , then  $\overset{\circ}{F}_\alpha \neq \emptyset$  is assumed, so that once again  $\partial F_\alpha = \partial \overset{\circ}{F}_\alpha$  and  $\partial F_\alpha$  is nowhere dense. On the other hand, it is trivial that  $x \in \partial F_\alpha$ , for if  $x \in \overset{\circ}{F}_\alpha$ , it follows from (5.1) that  $\bar{f}(x) \leq \alpha$ , which is a contradiction.

Since  $m \notin \mathbb{Q}_m$ , the above shows that  $\Sigma(\alpha)$  is always contained in a subset of  $X$  of first category (i.e.,  $F_\alpha$  or  $\partial F_\alpha$ ), so that it is of first category.  $\square$

It is not hard to check that  $\bar{f}$  in (5.1) is the usc hull of  $f$  (smallest usc function  $g \geq f$ ), but this is not relevant. In general,  $f \sim_{\mathcal{T}} \bar{f}$  is false: If  $f$  is a discontinuous linear form on  $X$ , then  $\overset{\circ}{F}_\alpha = \emptyset$  for every  $\alpha$ , so that  $\bar{f} = \infty \neq f$  everywhere.

Under the assumption made in part (iii) of Lemma 5.1,  $\bar{f}$  is real-valued and usc, so that its set of points of discontinuity is of first category. This is often quoted only when  $X$  is a complete metric space, but true and elementary in every topological space; see for instance [20, Lemma 2.1]. Even though  $f \sim_{\mathcal{T}} \bar{f}$ , the set of points of discontinuity of  $f$  need not be of first category: In Example 4.1,  $\bar{f} = g$  but  $f$  is nowhere continuous. However, Corollary 4.3 shows at once which condition is missing:

**Theorem 5.2.** *Let  $f : X \rightarrow \mathbb{R}$  be quasiconvex and set  $m := \mathcal{T} \operatorname{ess\,inf}_X f$ . Suppose also that (i)  $\overset{\circ}{F}_\alpha \neq \emptyset$  if  $\alpha > m$  and that (ii)  $f^{-1}(m) \cap \bar{F}_\alpha$  is of first category if  $m > -\infty$  and  $\alpha < m$ . Then, the set of points of discontinuity of  $f$  is of first category.*

*Proof.* Condition (i) is the same as in part (iii) of Lemma 5.1, so that, as noted before the theorem, the set of points of discontinuity of  $\bar{f}$  is of first category. Since also  $f \sim_{\mathcal{T}} \bar{f}$  and  $\bar{f}$  is real-valued and quasiconvex (Lemma 5.1), the set of points of discontinuity of  $f$  is of first category by Corollary 4.3 with  $g = \bar{f}$ .  $\square$

When  $X = \mathbb{R}^N$ , “first category”, “nowhere dense” and “empty interior” are synonymous for convex subsets. As a result, the hypotheses of Theorem 5.2 always hold since  $F_\alpha$  is of first (second) category when  $\alpha < m$  ( $\alpha > m$ ) by definition of  $m$ . Thus, the set of points of discontinuity of a real-valued quasiconvex function on  $\mathbb{R}^N$  is always of first category.

**Remark 5.1.** *Since the set of points of discontinuity of an lsc function is of first category, Theorem 5.2 when  $X = \mathbb{R}^N$  follows at once from the remark that the*

points of discontinuity of  $f$  are exactly those of its lsc hull<sup>3</sup>  $\underline{f}$  (largest lsc function  $g \leq f$ ). This is shown in [10, Proposition 3.5] and is false if  $\dim X = \infty$  (but see Theorem 6.4). Since  $\underline{f}$  is quasiconvex, this also implies that the  $\sigma$ -porosity result in [4, Theorem 19] in the lsc case actually applies equally to  $\underline{f}$  and  $f$  and so does not require lower semicontinuity.

Another quick proof of Theorem 5.2 when  $X = \mathbb{R}^N$  is that the set of points of discontinuity of any real-valued function  $f$  being an  $\mathcal{F}_\sigma$ , it can only be of first category or have nonempty interior. Since it has measure zero when  $f$  is quasiconvex ([6], [8]), the latter is impossible. Our attempts to use the  $\mathcal{F}_\sigma$  property to get a shorter proof of Theorem 5.2 were not successful when  $\dim X = \infty$ . However, it is noteworthy that it implies that if the set of points of discontinuity of a real-valued function  $f$  is not of first category, then  $f$  is discontinuous at *every* point of a nonempty open subset.

As a corollary, we obtain necessary and sufficient conditions for the set of points of discontinuity of  $f$  to be of first category.

**Corollary 5.3.** *Let  $f : X \rightarrow \mathbb{R}$  be quasiconvex and let  $m := \mathcal{T} \operatorname{ess\,inf}_X f$ . The following statements are equivalent:*

- (i) *The set of points of discontinuity of  $f$  is of first category.*
- (ii)  *$F_\alpha$  is nowhere dense whenever  $F'_\alpha = \emptyset$ .*
- (iii)  *$F_\alpha$  is nowhere dense whenever  $F'_\alpha = \emptyset, \alpha \neq m$ .*
- (iv)  *$F'_\alpha \neq \emptyset$  if  $\alpha > m$ ,  $F_\alpha$  is nowhere dense if  $m > -\infty$  and  $\alpha < m$ .*
- (v)  *$F'_\alpha \neq \emptyset$  if  $\alpha > m$ ,  $F_\alpha$  is nowhere dense if  $m > -\infty$  and  $\alpha < m$ .*
- (vi)  *$F'_\alpha \neq \emptyset$  if  $\alpha > m$ ,  $f^{-1}(m) \cap \overline{F}_\alpha$  is of first category if  $m > -\infty$  and  $\alpha < m$ .*

*Proof.* That (vi)  $\Rightarrow$  (i) follows from Theorem 5.2. Thus, it remains to show that (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (v)  $\Rightarrow$  (vi).

(i)  $\Rightarrow$  (ii). Assume that the set  $A$  of points of discontinuity of  $f$  is of first category. Observe that if  $f$  is continuous at  $x \in \overline{F}_\alpha$ , then  $x \in F'_\alpha$ .

Now, let  $\alpha \in \mathbb{R}$  be such that  $F_\alpha$  is *not* nowhere dense, so that  $\overline{F}_\alpha$  has nonempty interior  $U$ . If  $x \in U \setminus A$ , then  $f$  is continuous at  $x$  and so, from the above remark,  $U \setminus A \subset F'_\alpha$ . Since  $F'_\alpha$  is convex, it follows from Lemma 3.1 that  $U \subset F'_\alpha$  and hence that  $F'_\alpha \neq \emptyset$ .

Thus,  $F'_\alpha \neq \emptyset$  whenever  $F_\alpha$  is not nowhere dense, which is equivalent to saying that  $F_\alpha$  is nowhere dense whenever  $F'_\alpha = \emptyset$ . This proves (ii).

(ii)  $\Rightarrow$  (iii). Obvious.

(iii)  $\Rightarrow$  (iv). If  $\alpha > m$ , then  $F_\alpha$  is of second category by definition of  $m$ . It follows that  $F'_\alpha \neq \emptyset$ , for otherwise  $F_\alpha$  is nowhere dense by (iii), which is a contradiction. To see that  $F_\alpha$  is nowhere dense if  $m > -\infty$  and  $\alpha < m$ , observe that  $F'_\alpha \subset F'_\beta$  for any  $\beta \in (\alpha, m)$  and  $F_\beta$  is of first category by definition of  $m$ . Thus,  $F'_\alpha$  is of first category and so has empty interior. By (iii),  $F_\alpha$  is nowhere dense.

(iv)  $\Rightarrow$  (v). Since  $F'_\beta \subset F'_\alpha$  for every  $\beta < \alpha$ , it follows from (iv) that  $F'_\alpha \neq \emptyset$  when  $\alpha > m$ .

<sup>3</sup>The arctan trick can be used to make sure that  $\underline{f}$  is finite.

(v)  $\Rightarrow$  (vi) Obvious.  $\square$

On face value, (vi) is significantly weaker than (v), even though they are equivalent; see Theorem 6.2. Also, it is readily checked that  $F'_\alpha$  can be replaced by  $F_\alpha$  in (iii) (use (iii)  $\Rightarrow$  (v)) but this is not true in (ii): It is possible that the set of points of discontinuity of  $f$  is of first category and  $m > -\infty$ , yet  $F_m$  (always of first category, hence with empty interior) is not nowhere dense:

**Example 5.1.** *Let  $X$  be an infinite dimensional separable Banach space and let  $(x_n)_{n \in \mathbb{N}} \subset X$  be a dense sequence. Set  $K_n := \text{span}\{x_1, \dots, x_n\}$ , so that  $K := \bigcup_{n \in \mathbb{N}} K_n$  is a dense subspace of  $X$  and  $K$  is of first category since  $\dim K_n < \infty$ . After passing to a subsequence, we may assume  $K_n \subsetneq K_{n+1}$  without changing  $K$ . Now, let  $(\alpha_n) \subset \mathbb{R}$  be a strictly increasing sequence such that  $\lim_{n \rightarrow \infty} \alpha_n = 1$ . Set  $f(x) = 1$  if  $x \in X \setminus K$ ,  $f(x) = \alpha_1$  if  $x \in K_1$  and  $f(x) = \alpha_n$  if  $x \in K_n \setminus K_{n-1}$ ,  $n \geq 2$ . Since  $K$  is of first category,  $m = 1$ . If  $\alpha > 1$ , then  $F_\alpha = X$  has nonempty interior while  $F_1 = K$  is convex of first category, but everywhere dense. If  $\alpha_1 < \alpha < 1$ , there is a unique  $n \in \mathbb{N}$  such that  $\alpha_n < \alpha \leq \alpha_{n+1}$  and so  $F_\alpha = K_n$  is (convex and) nowhere dense. If  $\alpha \leq \alpha_1$ , then  $F_\alpha = \emptyset$ . This shows that  $f$  is quasiconvex and, by (i)  $\Leftrightarrow$  (v) in Corollary 5.3, that the set of points of discontinuity of  $f$  is of first category. (By a direct verification, this set is  $K$ .)*

Since  $\overset{\circ}{F}_\alpha \neq \emptyset$  or  $\overset{\circ}{F}'_\alpha \neq \emptyset$  is unchanged when  $\alpha$  is increased, Corollary 5.3 makes it clear that everything depends upon the behavior of  $f$  below the critical level  $m$  and “just above” it. In practice, this means the following:

**Corollary 5.4.** *If  $f : X \rightarrow \mathbb{R}$  is a quasiconvex function whose set of points of discontinuity is of first category and if  $g : X \rightarrow \mathbb{R}$  is another quasiconvex function such that  $G_\alpha := \{x \in X : g(x) < \alpha\}$  (or  $G'_\alpha := \{x \in X : g(x) \leq \alpha\}$ ) coincides with  $F_\alpha$  (or  $F'_\alpha$ ) for every  $\alpha < \alpha_0$  with  $\alpha_0 > m := \mathcal{T} \text{ess inf}_X f$ , then the set of points of discontinuity of  $g$  is of first category.*

*Proof.* Just notice that  $m = \mathcal{T} \text{ess inf}_X g$  and use the equivalence between (i) and (v) (or (iv)) for  $f$  and for  $g$  in Corollary 5.3.  $\square$

The hypotheses about  $g$  in Corollary 5.4 are equivalent to  $g = f$  on  $F_{\alpha_0}$  and  $f = g$  on  $G_{\alpha_0}$  and therefore stronger than either of these requirements alone. (Use  $f(x) = \inf\{\alpha \in \mathbb{R} : x \in F'_\alpha\}$  and likewise for  $g$ .) The points of discontinuity of  $f$  and  $g$  are of course the same inside  $F_{\alpha_0} = G_{\alpha_0}$  where  $f$  and  $g$  coincide, but unlike in Corollary 4.3,  $f$  and  $g$  may be completely different on large subsets, which makes the result rather unexpected. There are various obvious (and perhaps less obvious) ways to generalize Corollary 5.4.

## 6. SPECIAL CASES AND RELATED RESULTS

The first two theorems in this section give especially simple conditions ensuring that the set of points of discontinuity of a quasiconvex function is of first category. The first one is convenient for the proof of Theorem 6.3 further below.

**Theorem 6.1.** *Let  $f : X \rightarrow \mathbb{R}$  be quasiconvex and set  $m := \mathcal{T} \text{ess inf}_X f$ . If  $\overset{\circ}{F}'_\alpha \neq \emptyset$  (or  $\overset{\circ}{F}_\alpha \neq \emptyset$ ) when  $\alpha > m$  and  $\inf_X f = m$  (always true if  $m = -\infty$ ), the set of points of discontinuity of  $f$  is of first category.*

*Proof.* Since  $\inf_X f = m$  means  $F_\alpha = \emptyset$  when  $\alpha < m$ , this follows from the equivalence between (i) and (iv) (or (v)) in Corollary 5.3.  $\square$

**Example 6.1.** *If  $f : X \rightarrow \mathbb{R}$  is quasiconvex,  $f \geq 0$ ,  $f(0) = 0$  and  $f$  is continuous at 0, the set of points of continuity of  $f$  is of first category. Indeed, the continuity of  $f$  at 0 implies that  $F'_\alpha$  ( $F_\alpha$ ) is a neighborhood of 0 for every  $\alpha > 0$  and (hence) that  $m = 0$ .*

The first part of the next theorem follows at once from Corollary 5.3, but more information is obtained.

**Theorem 6.2.** *Let  $f : X \rightarrow \mathbb{R}$  be quasiconvex and set  $m := \mathcal{T} \operatorname{ess} \inf_X f$ . If  $\overset{\circ}{F}_\alpha \neq \emptyset$  when  $\alpha > m$  and either  $m = -\infty$  or  $m > -\infty$  and  $f^{-1}(m)$  is of first category, the set of points of discontinuity of  $f$  is of first category. Furthermore,  $F_m$  and  $f^{-1}(m)$  (and hence also  $F'_m$ ) are nowhere dense.*

*Proof.* Condition (vi) of Corollary 5.3 is trivially satisfied, which settles the issue about the points of discontinuity. To show that  $F_m$  is nowhere dense, observe that  $F'_m = f^{-1}(m) \cup F_m$  is of first category (since  $F_m$  is; see the proof of Corollary 4.3).

Thus,  $\overset{\circ}{F'_m} = \emptyset$  and the conclusion follows from the equivalence between (i) and (ii) in Corollary 5.3.

It remains to show that  $f^{-1}(m)$  is nowhere dense. Observe that if  $x \in \overline{f^{-1}(m)}$ , and  $f(x) \neq m$ , then  $f$  is not continuous at  $x$ . Thus,  $\overline{f^{-1}(m)} \subset f^{-1}(m) \cup A$ , where  $A$  is the set of points of discontinuity of  $f$ . Since  $A$  is of first category, it follows that  $\overline{f^{-1}(m)}$  is of first category. Therefore, it has empty interior and  $f^{-1}(m)$  is nowhere dense.  $\square$

If a function  $f : X \rightarrow \mathbb{R}$  is continuous, its level sets  $f^{-1}(\alpha)$  are closed. As a result, either they have nonempty interior or they are nowhere dense. Theorem 6.3 below shows that, with at most one exception, the same thing is true for quasiconvex functions whose set of points of discontinuity is of first category. The proof is based primarily on Theorem 6.1. In that regard, see Remark 6.1.

**Theorem 6.3.** *Suppose that  $f : X \rightarrow \mathbb{R}$  is quasiconvex and that the set of points of discontinuity of  $f$  is of first category. Set  $m := \mathcal{T} \operatorname{ess} \inf_X f$ . If  $\alpha \in \mathbb{R}$  and  $\alpha \neq m$  (always true if  $m = -\infty$ ), then  $f^{-1}(\alpha)$  is nowhere dense or has nonempty interior.*

*Proof.* If  $m > -\infty$  and  $\alpha < m$ , then  $f^{-1}(\alpha) \subset F_\beta$  for any  $\beta \in (\alpha, m)$ , whence  $f^{-1}(\alpha)$  is nowhere dense by the equivalence between (i) and (v) of Corollary 5.3. Thus, from now on,  $\alpha > m$  and it is assumed that  $f^{-1}(\alpha)$  has empty interior. The problem is to show that  $f^{-1}(\alpha)$  is actually nowhere dense.

In the proof of Theorem 6.2, the fact that  $f^{-1}(m)$  is nowhere dense if it is of first category depends only on the set of points of discontinuity of  $f$  being of first category. Thus, the same thing is true when  $m$  is replaced by any other value  $\alpha$  as soon as  $f^{-1}(\alpha)$  is of first category. Accordingly, it suffices to show that  $f^{-1}(\alpha)$  is of first category when  $\alpha > m$  and  $f^{-1}(\alpha)$  has empty interior. This is done below.

If  $h : \mathbb{R} \rightarrow \mathbb{R}$  is nondecreasing,  $h \circ f$  is quasiconvex. In particular, this is true of  $h^-_\alpha \circ f$  and  $h^+_\alpha \circ f$ , where  $h^-_\alpha(t) := 0$  if  $0 \leq t \leq \alpha$ ,  $h^-_\alpha(t) := 1$  if  $t > \alpha$  and  $h^+_\alpha(t) := 0$  if  $0 \leq t < \alpha$ ,  $h^+_\alpha(t) := 1$  if  $t \geq \alpha$ . Below, we prove that the sets of points of discontinuity of  $h^-_\alpha \circ f$  and  $h^+_\alpha \circ f$  are of first category and, next, that if  $x \in f^{-1}(\alpha)$ , at least one among  $h^-_\alpha \circ f$  and  $h^+_\alpha \circ f$  is not continuous at  $x$ . If

so,  $f^{-1}(\alpha)$  is contained in the union of two sets of first category and the proof is complete.

*Step 1:* The set of points of discontinuity of  $h_\alpha^- \circ f$  is of first category.

If  $a > 0$ , the set  $\Phi_a := \{x \in X : h_\alpha^- \circ f < a\}$  contains the set  $F_\alpha$  and so it has nonempty interior since  $F_\alpha \neq \emptyset$  by the equivalence of (i) and (v) of Corollary 5.3 (recall  $\alpha > m$ ). In particular,  $\Phi_a$  is of second category, whence  $\mathcal{T} \text{ess inf}_X(h_\alpha^- \circ f) \leq a$ . Since  $a > 0$  is arbitrary,  $\mathcal{T} \text{ess inf}_X(h_\alpha^- \circ f) \leq 0$ . Thus,  $\mathcal{T} \text{ess inf}_X(h_\alpha^- \circ f) = 0$  since  $h_\alpha^- \circ f \geq 0$ . Furthermore,  $\inf_X(h_\alpha^- \circ f) = 0$  since  $0 \leq \inf_X(h_\alpha^- \circ f) \leq \mathcal{T} \text{ess inf}_X(h_\alpha^- \circ f) = 0$ . It follows that Theorem 6.1 is applicable to  $h_\alpha^- \circ f$ . This completes Step 1.

*Step 2:* The set of points of discontinuity of  $h_\alpha^+ \circ f$  is of first category.

All the arguments of Step 1 can be repeated verbatim.

*Step 3:* If  $x \in f^{-1}(\alpha)$ , then either  $h_\alpha^- \circ f$  or  $h_\alpha^+ \circ f$  is not continuous at  $x$ .

By contradiction, if both functions are continuous at  $x$ , there is a neighborhood  $U$  of  $x$  such that  $U \subset (h_\alpha^- \circ f)^{-1}(-1/2, 1/2)$  (because  $h_\alpha^-(\alpha) = 0$ ) and  $U \subset (h_\alpha^+ \circ f)^{-1}(1/2, 3/2)$  (because  $h_\alpha^+(\alpha) = 1$ ). Then,  $h_\alpha^-(f(y)) = 0$  and  $h_\alpha^+(f(y)) = 1$  for every  $y \in U$ , i.e.,  $f(y) \leq \alpha$  and  $f(y) \geq \alpha$ , respectively, for every  $y \in U$ . Therefore  $f = \alpha$  on  $U$ , which contradicts the assumption that  $f^{-1}(\alpha)$  has empty interior.  $\square$

**Remark 6.1.** *In Theorem 6.3, suppose that  $f$  is lsc (usc). Then,  $h_\alpha^+ \circ f$  ( $h_\alpha^- \circ f$ ) in the proof need not be lsc or usc. Therefore, even when  $f$  is lsc or usc, the theorem does not follow from the fact that the set of points of discontinuity of an lsc or usc function is of first category.*

If  $f$  is a discontinuous linear form on  $X$ , then  $f^{-1}(\alpha)$  has empty interior but is everywhere dense for every  $\alpha \in \mathbb{R}$ . In Example 4.1,  $m = 1$  and  $f^{-1}(0) = H$  is of first category but also everywhere dense. Thus, Theorem 6.3 breaks down quite easily when the set of points of discontinuity of  $f$  is not of first category.

If  $X = \mathbb{R}^N$ , then  $\alpha = m$  need not be ruled out in Theorem 6.3 because the set of points of discontinuity of a quasiconvex function on  $\mathbb{R}^N$  is always of first category (so, Step 1 and Step 2 of the proof are unnecessary irrespective of  $\alpha$ ). However,  $f^{-1}(m)$  may be of second category with empty interior when  $\dim X = \infty$ ; see Example 5.1.

**Remark 6.2.** *By using the  $\sigma$ -porosity result in [4, Theorem 19], duly extended to the non lsc case (Remark 5.1) in Step 3 of proof of Theorem 6.3 but now with arbitrary  $\alpha$  (everything else can be ignored), it follows that if  $X = \mathbb{R}^N$ , every level set  $f^{-1}(\alpha)$  with empty interior is  $\sigma$ -porous.*

If  $X$  is separable,  $f^{-1}(\alpha)$  in Theorem 6.3 cannot have nonempty interior for more than countably many values  $\alpha \in \mathbb{R}$  since the sets  $f^{-1}(\alpha)$  are disjoint. Elementary one-dimensional examples show that countably many level sets  $f^{-1}(\alpha)$  may indeed have nonempty interior.

In Remark 5.1, we pointed out that a quasiconvex function on  $\mathbb{R}^N$  has the same points of (dis)continuity as its lsc hull  $\underline{f}$  but that this is generally false when  $\mathbb{R}^N$  is replaced by an infinite dimensional tvs  $\bar{X}$ . Below, we show that this is still true if the set of points of discontinuity of  $f$  is of first category, but first a minor technicality must be clarified: Even when  $f$  is real-valued, it may happen that  $\underline{f} = -\infty$  at some points. This is the reason why the actual statement is a little more technical than the short version just mentioned.

**Theorem 6.4.** *Let  $f : X \rightarrow \mathbb{R}$  be quasiconvex and let  $\underline{f}$  denote its lsc hull. If the set  $A$  of points of discontinuity of  $f$  is of first category, then  $A = \{x \in X : \underline{f}(x) = -\infty\} \cup \{x \in X : \underline{f}(x) \in \mathbb{R}, \underline{f} \text{ is discontinuous at } x\}$ .*

*Proof.* A routine verification shows that

$$(6.1) \quad \underline{f}(x) = \inf\{\alpha \in \mathbb{R} : x \in \overline{F}_\alpha\}.$$

We shall prove that  $f$  is continuous at  $x$  if and only if  $\underline{f}(x) \in \mathbb{R}$  and  $\underline{f}$  is continuous at  $x$ , which is equivalent to the claim made in the theorem.

*Step 1:* Assume that  $f$  is continuous at  $x$ .

Given  $\varepsilon > 0$ , set  $I_\varepsilon : (f(x) - \varepsilon, f(x) + \varepsilon)$ , so that there is an open neighborhood  $W_\varepsilon$  of  $x$  such that  $f(W_\varepsilon) \subset I_\varepsilon$ . Since  $\underline{f} \leq f$ , it is obvious that  $\underline{f}(y) < f(x) + \varepsilon$  for every  $y \in W_\varepsilon$ . We claim that  $\underline{f}(y) \geq f(x) - \varepsilon$  for every  $y \in W_\varepsilon$ . Otherwise, there is  $y \in W_\varepsilon$  such that  $\underline{f}(y) < f(x) - \varepsilon$  and so there is  $\alpha < f(x) - \varepsilon$  such that  $y \in \overline{F}_\alpha$ . If so,  $W_\varepsilon \cap F_\alpha \neq \emptyset$  since  $W_\varepsilon$  is a neighborhood of  $y$ , which is absurd since  $\alpha < f(x) - \varepsilon$  and  $f(W_\varepsilon) \subset I_\varepsilon$ . Thus,  $\underline{f}(W_\varepsilon) \subset [f(x) - \varepsilon, f(x) + \varepsilon)$ , which show that  $\underline{f}(x) \in \mathbb{R}$  (even  $\underline{f}(x) = f(x)$ ) and that  $\underline{f}$  is continuous at  $x$ .

*Step 2:* Assume  $\underline{f}(x) \in \mathbb{R}$  and  $\underline{f}$  is continuous at  $x$ .

Let  $\beta > \underline{f}(x)$  be given, so that  $W := \underline{f}^{-1}((-\infty, \beta))$  is a neighborhood of  $x$ . By definition of  $\underline{f}$  in (6.1),  $W \subset \overline{F}_\beta$  and so  $\overline{F}_\beta$  has nonempty interior containing  $x$ . Note also that if  $y \in \overline{F}_\beta$  and  $y \notin F'_\beta$ , then  $f$  is not continuous at  $y$ , so that  $\overline{F}_\beta \subset F'_\beta \cup A$  where  $A$  is the set of points of discontinuity of  $f$ . Altogether, this yields  $\overset{\circ}{\overline{F}_\beta} \subset F'_\beta \cup A$ . Since  $A$  is of first category, then  $B := A \cap (X \setminus F'_\beta)$  is of first category. Therefore,  $\overset{\circ}{\overline{F}_\beta} \setminus B \subset F'_\beta$  and so, since  $F'_\beta$  is convex, it follows from Lemma 3.1 that  $\overset{\circ}{\overline{F}_\beta} \subset F'_\beta$ . Since  $x \in \overset{\circ}{\overline{F}_\beta}$ , this shows that  $F'_\beta$  is a neighborhood of  $x$  for every  $\beta > \underline{f}(x)$  and (hence) that  $f(x) = \underline{f}(x)$  since  $\underline{f} \leq f$ .

In summary,  $F'_\beta$  is a neighborhood of  $x$  for every  $\beta > f(x)$ . On the other hand, since  $\underline{f} \leq f$  and  $\underline{f}$  is continuous at  $x$ , it follows that  $f^{-1}([\alpha, \infty)) \supset \underline{f}^{-1}([\alpha, \infty))$ , a neighborhood of  $\underline{f}(x) = f(x)$  for every  $\alpha > f(x)$ . Altogether,  $f^{-1}([\alpha, \beta])$  is a neighborhood of  $x$  for every  $\alpha < f(x) < \beta$ , whence  $f$  is continuous at  $x$ .  $\square$

## 7. COMPLEMENTS

We complete this paper with two complementary results of independent interest. The first one clarifies the connection between the conditions for the set of points of discontinuity of a quasiconvex function to be first category (1) knowing that this is true for an equivalent quasiconvex function (Corollary 4.3) and (2) without knowing that this is true for an equivalent quasiconvex function (Sections 5 and 6).

The second result was motivated by the conditions (ii) and (ii') of Lemma 2.1, that suggest a connection between quasiconvex and quasicontinuous functions.

The sets  $\overset{\circ}{F}_\alpha$  or  $\overset{\circ}{F}'_\alpha$  are involved repeatedly in Sections 5 and 6. For example, in Corollary 5.3, it is shown that the set of points of discontinuity of  $f$  is of first category if and only if  $F_\alpha$  is nowhere dense whenever  $\overset{\circ}{F}'_\alpha = \emptyset$ . Assuming this, Corollary 4.3 gives a necessary and sufficient condition for the set of points of continuity of another quasiconvex function  $g \sim_{\mathcal{T}} f$  to be of first category (after exchanging the roles of  $f$  and  $g$  in the notation), but this condition makes no

implicit or explicit reference to  $\overset{\circ}{F}'_\alpha$  or to its analog for  $g$ . Yet, in a self-explanatory notation, it must somehow imply that  $G_\alpha$  is nowhere dense whenever  $\overset{\circ}{G}'_\alpha = \emptyset$ .

This is clarified in Theorem 7.2 below, by showing that  $\overset{\circ}{F}_\alpha$  and  $\overset{\circ}{F}'_\alpha$  are unchanged after  $f$  is replaced by any equivalent quasiconvex function. To see this, we need a variant of Lemma 3.1.

**Lemma 7.1.** *Let  $U \subset X$  be open and convex and let  $A_1, A_2 \subset X$  be of first category with  $A_2 \cap U = \emptyset$ . If  $G := (U \setminus A_1) \cup A_2$  is convex, then  $A_1 \cap U = \emptyset$  and  $\overset{\circ}{G} = U$ .*

*Proof.* (i) The result is trivial if  $U = \emptyset$ , for then  $G = A_2$  is of first category, so that  $\overset{\circ}{G} = \emptyset$ . From now on,  $U \neq \emptyset$ . We first prove that if  $A_1 \cap U \neq \emptyset$ , there are  $x, y \in U \setminus A_1 \subset G$  such that  $[x, y] \subset U$  contains a point  $z \in A_1$ . Since  $A_2 \cap (U \setminus A_1) \subset A_2 \cap U = \emptyset$ , it follows that  $z \notin G$ , which contradicts the convexity of  $G$ .

To prove the claim, suppose that  $A_1 \cap U \neq \emptyset$ . After translation, it is not restrictive to assume  $0 \in A_1 \cap U$ . Thus,  $0 \in A_1 \cup (-A_1)$ , a set of first category. The set  $V := U \cap (-U) \subset U$  is an open neighborhood of 0. Since  $X$  is a Baire space, there is a point  $x \in V \setminus (A_1 \cup (-A_1))$ . Equivalently, both  $x$  and  $-x$  are in  $V \setminus A_1 \subset U \setminus A_1$  and their midpoint 0 is in  $A_1$ . This proves the claim with  $y = -x$ .

At this stage, we have shown that  $A_1 \cap U = \emptyset$ , so that  $U \setminus A_1 = U$  and  $G = U \cup A_2$ . Thus,  $U \subset \overset{\circ}{G}$ . Also,  $\overset{\circ}{G} \subset G$  and so  $\overset{\circ}{G} = U \cup A_3$  with  $A_3 = A_2 \cap \overset{\circ}{G}$  of first category. Since  $A_2 \cap U = \emptyset$ , hence  $A_3 \cap U = \emptyset$ , this also reads  $U = \overset{\circ}{G} \setminus A_3$ . Therefore, by changing  $U$  into  $\overset{\circ}{G}$  (convex),  $G$  into  $U$ ,  $A_1$  into  $A_3$  and  $A_2$  into  $\emptyset$  above, it follows that  $\overset{\circ}{G} \subset U$  and so  $\overset{\circ}{G} = U$ .  $\square$

**Theorem 7.2.** *If  $f : X \rightarrow \overline{\mathbb{R}}$  is quasiconvex, then  $\overset{\circ}{F}_\alpha$  and  $\overset{\circ}{F}'_\alpha$  are unchanged when  $f$  is replaced by another quasiconvex function  $g \sim_{\mathcal{T}} f$ .*

*Proof.* We give the proof for  $\overset{\circ}{F}_\alpha$ . Call  $G_\alpha$  the set  $\{x \in X : g(x) < \alpha\}$ . Below, we prove that if  $\overset{\circ}{F}_\alpha \neq \emptyset$ , then  $G_\alpha = (\overset{\circ}{F}_\alpha \setminus A_1) \cup A_2$  where  $A_1$  and  $A_2$  are of first category and  $A_2 \cap \overset{\circ}{F}_\alpha = \emptyset$ . From Lemma 7.1, this implies  $\overset{\circ}{G}_\alpha = \overset{\circ}{F}_\alpha$  and, by exchanging the roles of  $f$  and  $g$ , it follows that  $\overset{\circ}{G}_\alpha = \overset{\circ}{F}_\alpha$  also when  $\overset{\circ}{G}_\alpha \neq \emptyset$ . But then,  $\overset{\circ}{G}_\alpha = \overset{\circ}{F}_\alpha$  irrespective of whether either is nonempty.

First, since  $g \sim_{\mathcal{T}} f$ , it is clear that  $F_\alpha = (G_\alpha \setminus B_1) \cup A_1$  where  $A_1$  and  $B_1$  are of first category,  $B_1 \subset G_\alpha$  and  $A_1 \cap G_\alpha = \emptyset$ . For future use, note that  $B_1 \cap F_\alpha = \emptyset$  since  $A_1 \cap B_1 \subset A_1 \cap G_\alpha = \emptyset$ . Next, since  $\overset{\circ}{F}_\alpha \neq \emptyset$ , observe that  $F_\alpha = \overset{\circ}{F}_\alpha \cup B_2$  where  $B_2 \subset \partial F_\alpha = \partial \overset{\circ}{F}_\alpha$  (Remark 3.1) is nowhere dense. Obviously,  $B_2 \cap \overset{\circ}{F}_\alpha = \emptyset$ .

As a result,  $(G_\alpha \setminus B_1) \cup A_1 = \overset{\circ}{F}_\alpha \cup B_2$ , both sides being  $F_\alpha$ . Since  $A_1 \cap (G_\alpha \setminus B_1) \subset A_1 \cap G_\alpha = \emptyset$ , we may take out  $A_1$  from both sides without removing any point of  $G_\alpha \setminus B_1$  and so  $G_\alpha \setminus B_1 = (\overset{\circ}{F}_\alpha \cup B_2) \setminus A_1 = (\overset{\circ}{F}_\alpha \setminus A_1) \cup (B_2 \setminus A_1)$ . Now, since  $B_1 \subset G_\alpha$ , it follows that  $G_\alpha = (\overset{\circ}{F}_\alpha \setminus A_1) \cup A_2$  where  $A_2 := (B_2 \setminus A_1) \cup B_1$ . Both  $A_1$  and  $A_2$  are of first category and  $A_2 \cap \overset{\circ}{F}_\alpha = \emptyset$  since  $B_2 \cap \overset{\circ}{F}_\alpha = \emptyset$  and  $B_1 \cap F_\alpha = \emptyset$ .  $\square$



If “second category” is replaced by “nonempty interior” in condition (ii) ((ii')) of Lemma 2.1, the modified condition is exactly the assumption that the function  $f$  of interest is what is called *lower quasicontinuous* (*upper quasicontinuous*). See [11], [21], [22], among others.

The stronger requirement that for every open subset  $U \subset X$  and every open subset  $\Omega \subset \mathbb{R}$  such that  $f(U) \cap \Omega \neq \emptyset$ , the set  $U \cap f^{-1}(\Omega)$  has nonempty interior is called *quasicontinuity*, a concept introduced by Kempisty [17] in 1932, which has been vigorously revisited in recent past. It has nothing to do with its namesake occasionally used in convex analysis ([16], [19]) and makes sense when the target space  $\mathbb{R}$  is replaced by any topological space  $Y$ .

A well known shorter equivalent definition is that  $f$  is quasicontinuous if and only if, for every open subset  $\Omega \subset \mathbb{R}$ , the interior of  $f^{-1}(\Omega)$  is dense in  $f^{-1}(\Omega)$ . Shi *et al.* [26] used this definition to rediscover quasicontinuous functions six decades later under the name “robust functions”, but this terminology is not prevalent.

Every continuous function is quasicontinuous but quasicontinuity and semicontinuity are very different properties: With  $X = \mathbb{R}$ , the function  $f = \chi_{\mathbb{R} \setminus \{0\}}$  is lsc but not quasicontinuous, whereas  $f(x) = \sin(x^{-1})$  if  $x \neq 0$  and  $f(0) = 0$  is quasicontinuous but neither lsc nor usc.

At any rate, the resemblance noted in Lemma 2.1 raises the natural question whether a real-valued quasiconvex function is quasicontinuous. The answer is positive in the usc case, as we now show. The Baire property is not needed.

**Theorem 7.3.** *Let  $X$  be a tvs. If  $f : X \rightarrow \mathbb{R}$  is quasiconvex and usc, then  $f$  is quasicontinuous.*

*Proof.* Let  $\Omega \subset \mathbb{R}$  be open. It must be shown that the interior of  $f^{-1}(\Omega)$  is dense in  $f^{-1}(\Omega)$ . A routine verification reveals that this is true if and only if it is true when  $\Omega = (\alpha, \beta)$  is a finite interval such that  $f^{-1}(\Omega) \neq \emptyset$ .

It is plain that  $f^{-1}(\Omega) = F_\beta \setminus F'_\alpha$ . Since  $f$  is usc and quasiconvex,  $F_\beta$  is open and convex and, by the openness of  $F_\beta$ , the interior of  $f^{-1}(\Omega)$  is  $F_\beta \setminus \overset{\circ}{F'_\alpha}$ . Let  $W$  denote the interior of  $\overset{\circ}{F'_\alpha}$ . We begin by showing that  $W = \overset{\circ}{F'_\alpha}$ . Since this is obvious if  $W = \emptyset$ , we henceforth assume that  $W \neq \emptyset$ . If  $\dim X = \infty$ , it is generally false that a convex subset has the same interior as its closure (this does *not* follow from Remark 3.1), so a proof is needed. That  $f$  is usc is important.

First,  $F'_\alpha = \cap_{\gamma > \alpha} F_\gamma$ , so that  $\overset{\circ}{F'_\alpha} \subset \cap_{\gamma > \alpha} \overset{\circ}{F'_\gamma}$ . It follows that  $W \subset \overset{\circ}{F'_\gamma}$  for every  $\gamma > \alpha$ . Furthermore,  $F_\gamma$  (open since  $f$  is usc) is the interior of  $\overset{\circ}{F'_\gamma}$  (Remark 3.1) and so  $W \subset F_\gamma$ . As a result,  $W \subset \cap_{\gamma > \alpha} F_\gamma = F'_\alpha$ , so that  $W$  is contained in  $\overset{\circ}{F'_\alpha}$ . By definition of  $W$ , the converse is trivial, so that  $W = \overset{\circ}{F'_\alpha}$ , as claimed.

If  $x \in F_\beta \setminus \overset{\circ}{F'_\alpha}$  ( $= f^{-1}(\Omega) \neq \emptyset$ ), then  $x \notin \overset{\circ}{F'_\alpha} = W$ . Since  $W$  is the interior of  $\overset{\circ}{F'_\alpha}$ , every open neighborhood of  $x$  in  $X$  contains a point not in  $\overset{\circ}{F'_\alpha}$ . In particular, if  $U$  is an open neighborhood of  $x$  in  $X$ , then  $U \cap F_\beta$  (open) contains a point  $y \notin \overset{\circ}{F'_\alpha}$ . Hence,  $U$  contains a point of  $F_\beta \setminus \overset{\circ}{F'_\alpha}$ , so that  $F_\beta \setminus \overset{\circ}{F'_\alpha}$  is dense in  $F_\beta \setminus F'_\alpha$ .  $\square$

The example when  $X = \mathbb{R}$ ,  $f(0) = -1$  and  $f(x) = |x|$  when  $x \neq 0$  shows that  $f$  need not be quasicontinuous if it is not usc.

## REFERENCES

- [1] Arias de Reyna, J. Dense hyperplanes of first category, *Math. Annalen* **249** (1980) 111-114.
- [2] Aussel, D. and Daniilidis, A., Normal characterization of the main classes of quasiconvex functions, *Set-Valued Anal.* **8** (2000) 219-236.
- [3] Berberian, S. K., *Lectures in functional analysis and operator theory*, Graduate Texts in Mathematics, Vol. 15, Springer-Verlag, New York, 1974.
- [4] Borwein, J. M. and Wang, X., Cone-Monotone Functions: Differentiability and Continuity, *Canad. J. Math.* **57** (2005) 961-982.
- [5] Bourbaki, N., *Eléments de mathématique XV*, Actualités Sci. Ind., no. 1189, Herman, Paris, 1953.
- [6] Chabrilac, Y. and Crouzeix, J.-P., Continuity and differentiability properties of monotone real functions of several variables, *Math. Programming Study* **30** (1987) 1-16.
- [7] Conway, J. B., *A course in functional analysis*, Graduate Texts in Mathematics, Vol. 96, Springer-Verlag, New York, 1985.
- [8] Crouzeix, J.-P., Some differentiability properties of quasiconvex functions on  $\mathfrak{R}^n$ , *Optimization and Optimal Control*, A. Auslender, W. Oettli and J. Stoer, editors, Springer-Verlag, Lecture Notes in Control and Information Sciences, Vol. 30 (1981) 9-20.
- [9] Crouzeix, J.-P., Continuity and differentiability properties of quasiconvex functions on  $\mathbb{R}^n$ , in: *Generalized Concavity in Optimization and Economics*, S. Schaible and W.T. Ziemba editors, Academic Press (1981) 109-130.
- [10] Crouzeix, J.-P., Continuity and differentiability of quasiconvex functions, *Handbook of generalized convexity and generalized monotonicity*, Nonconvex Optim. Appl., Vol. 76, Springer-Verlag (2005) 121-149.
- [11] Ewert, J. and Lipski, T., Lower and upper quasicontinuous functions, *Demonstratio Math.* **16** (1983) 85-93.
- [12] Ger, R. and Kominek, Z., Boundedness and continuity of additive and convex functionals, *Aequationes Math.* **37** (1989) 252-258.
- [13] Goffman, C., Neugebauer, C. J. and Nishiura, T., Density topology and approximate continuity, *Duke Math. J.* **28** (1961) 497-505.
- [14] Hadjisavvas, N., Continuity properties of quasiconvex functions in infinite-dimensional spaces. Working Paper No.94-03, Graduate School of Management, Univ. of California, April 1994.
- [15] Haupt, O. and Pauc, C., La topologie approximative de Denjoy envisagée comme vraie topologie, *C. R. Acad. Sci. Paris* **234** (1952) 390-392.
- [16] Joly, J.-L. and Laurent, P.-J., Stability and duality in convex minimization problems, *Rev. Française Informat. Recherche Opérationnelle* **5** (1971) 3-42.
- [17] Kempisty, S., Sur les fonctions quasicontinues, *Fund. Math.* **19** (1932) 184-197.
- [18] Lukeš, J., Malý, J. and Zajíček, L., *Fine topology methods in real analysis and potential theory*, Lecture Notes in Mathematics **1189**, Springer-Verlag, Berlin, 1986.
- [19] Moussaoui, M. and Volle, M., Quasicontinuity and united functions in convex duality theory, *Comm. Appl. Nonlinear Anal.* **4** (1997) 73-89.
- [20] Namioka, I., Right topological groups, distal flows, and a fixed-point theorem, *Math. Systems Theory* **6** (1972) 193-209.
- [21] Neubrunn, T., Quasi-continuity, *Real Anal. Exchange* **14** (1988-89) 259-307.
- [22] Sakálová, K., graph continuity and quasicontinuity, *Tatra Mt. Math. Publ.* **2** (1993) 69-75.
- [23] Saxon, S. A., Non-Baire hyperplanes in nonseparable Baire spaces, *J. Math. Anal. Appl.* **168** (1992) 460-468.
- [24] Semadeni, Z., Spaces of continuous functions (VI) (Localization of multiplicative linear functionals), *Studia Math.* **23** (1963) 59-84.
- [25] Semadeni, Z., Functions with sets of points of discontinuity belonging to a fixed ideal, *Fund. Math.* **52** (1963) 25-39.
- [26] Shi, S. Z., Zheng, Q. and Zhuang, D., Discontinuous robust mappings are approximatable, *Trans. Amer. Math. Soc.* **347** (1995) 4943-4957.
- [27] Valdivia, M., *Topics in locally convex spaces*, North-Holland Mathematics Studies Vol. 67, North-Holland, Amsterdam 1982.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PITTSBURGH, PITTSBURGH, PA 15260  
 E-mail address: rabier@imap.pitt.edu